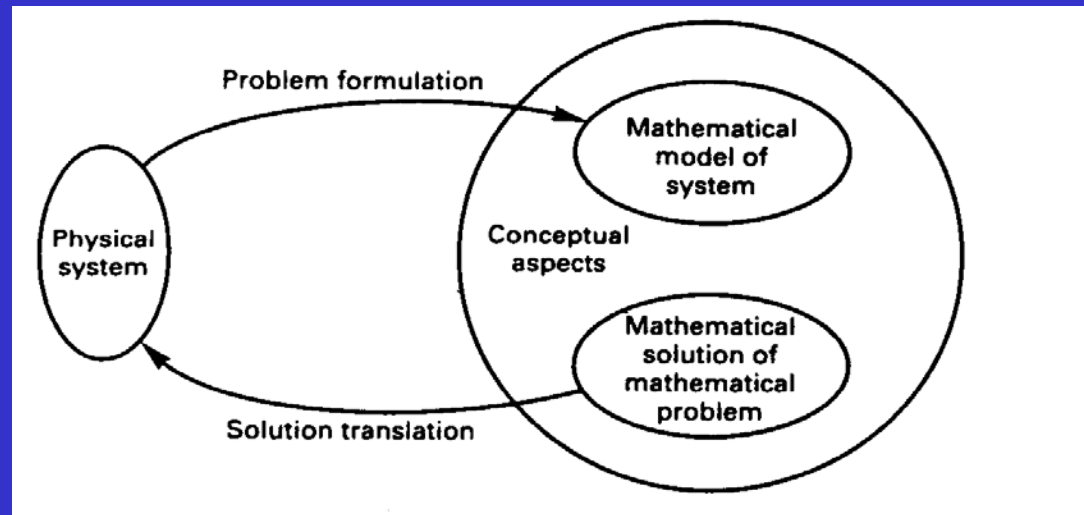


Mathematical Models of Physical Systems

- Why mathematical models of physical systems needed?
- Design of engineering systems by trying and error versus design by using mathematical models.
- Physical laws such as Newton's second law of motion is a mathematical model.
- Mathematical model gives the mathematical relationships relating the output of a system to its input.

Mathematical Models of Physical Systems (continued)

- Control systems give desired output by controlling the input. Therefore control systems and mathematical modeling are inter-linked.



- Mathematical models of control systems are ordinary differential equations
- To deal with linear o.d.e, Laplace Transform is an efficient approach.

What is Laplace Transform?

- The Laplace transform of a function $f(t)$ is defined as:

$$F(s) = \ell[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

- The inverse Laplace transform is defined as:

$$f(t) = \ell^{-1}[F(s)] = \frac{1}{2j\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds$$

where $j = \sqrt{-1}$ and the value of σ is determined by the singularities of $F(s)$.

$$s \equiv \frac{d}{dt}, \quad \frac{1}{s} \equiv \int_0^t dt$$

Why Is Laplace Transform Useful ?

- Model a linear time-invariant analog system as an algebraic system (transfer function).
- In control theory, Laplace transform converts linear differential equations into algebraic equations.
- This is much easier to manipulate and analyze.

An Example

- The Laplace transform of e^{-at} can be obtained by:

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(a+s)t} dt = \left. \frac{-e^{-(s+a)t}}{s+a} \right|_0^{\infty} = \frac{1}{s+a}$$

Linearity property

- These are useful properties:

$$\ell[kf(t)] = k\ell[f(t)] = kF(s)$$

$$\ell[f_1(t) + f_2(t)] = \ell[f_1(t)] + \ell[f_2(t)] = F_1(s) + F_2(s)$$

| Name | Time function f(t) | Laplace Transform |
|-----------------------|--------------------|-------------------------|
| Unit Impulse | $\delta(t)$ | 1 |
| Unit Step | $u(t)$ | $1/s$ |
| Unit ramp | t | $1/s^2$ |
| nth-Order ramp | t^n | $n!/s^{n+1}$ |
| Exponential | e^{-at} | $1/(s+a)$ |
| nth-Order exponential | $t^n e^{-at}$ | $n!/(s+a)^{n+1}$ |
| Sine | $\sin(bt)$ | $b/(s^2+b^2)$ |
| Cosine | $\cos(bt)$ | $s/(s^2+b^2)$ |
| Damped sine | $e^{-at} \sin(bt)$ | $b/((s+a)^2+b^2)$ |
| Damped cosine | $e^{-at} \cos(bt)$ | $(s+a)/((s+a)^2+b^2)$ |
| Diverging sine | $t \sin(bt)$ | $2bs/(s^2+b^2)^2$ |
| Diverging cosine | $t \cos(bt)$ | $(s^2-b^2)/(s^2+b^2)^2$ |

TABLE 2.2 Laplace transform theorems

| Item no. | Theorem | Name |
|----------|--|------------------------------------|
| 1. | $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$ | Definition |
| 2. | $\mathcal{L}[kf(t)] = kF(s)$ | Linearity theorem |
| 3. | $\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$ | Linearity theorem |
| 4. | $\mathcal{L}[e^{-at} f(t)] = F(s + a)$ | Frequency shift theorem |
| 5. | $\mathcal{L}[f(t - T)] = e^{-sT} F(s)$ | Time shift theorem |
| 6. | $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$ | Scaling theorem |
| 7. | $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$ | Differentiation theorem |
| 8. | $\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$ | Differentiation theorem |
| 9. | $\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0-)$ | Differentiation theorem |
| 10. | $\mathcal{L}\left[\int_{0-}^1 f(\tau) d\tau\right] = \frac{F(s)}{s}$ | Integration theorem |
| 11. | $f(\infty) = \lim_{s \rightarrow 0} sF(s)$ | Final value theorem ¹ |
| 12. | $f(0+) = \lim_{s \rightarrow \infty} sF(s)$ | Initial value theorem ² |

Find the Laplace transform of $f(t)=5u(t)+3e^{-2t}$.

• Solution:

$$\ell[5u(t)] = 5\ell[u(t)] = \frac{5}{s}$$

$$\ell[3e^{-2t}] = 3\ell[e^{-2t}] = \frac{3}{s+2}$$

$$F(s) = \frac{5}{s} + \frac{3}{s+2} = \frac{8s+10}{s(s+2)}$$

Another Example :

$$\frac{d^2 y}{dt^2} + 12 \frac{dy}{dt} + 32y = 32u(t)$$

Partial Fraction Expansion

(Case 1: Roots of Denominator are Real and distinct)

Find the inverse Laplace transform of

$$F(s)=5/(s^2+3s+2).$$

Solution:

$$F(s) = \frac{5}{s^2 + 3s + 2} = \frac{k_1}{s + 1} + \frac{k_2}{s + 2}$$

$$k_1 = (s + 1) F(s) \big|_{s=-1} = \frac{5}{s + 2} \big|_{s=-1} = 5$$

$$k_2 = (s + 2) F(s) \big|_{s=-2} = \frac{5}{s + 1} \big|_{s=-2} = -5$$

$$F(s) = \frac{5}{s + 1} + \frac{-5}{s + 2}$$

So we have :

$$\ell^{-1}[F(s)] = f(t) = (5e^{-t} - 5e^{-2t})u(t)$$

Exercise: Do example 2.3 of the textbook

Laplace Transform solution of a differential equation

$$\frac{d^2 y}{dt^2} + 12 \frac{dy}{dt} + 32y = 32u(t)$$

Case 2: Roots of the Denominator are Real and Repeated

$$F(s) = \frac{N(s)}{(s + p_1)^r (s + p_2)} \quad \text{Note that roots are } -p_i$$

$$F(s) = \frac{k_1}{(s + p_1)^r} + \frac{k_2}{(s + p_1)^{r-1}} + \dots + \frac{k_r}{(s + p_1)} + \frac{k_{r+1}}{(s + p_2)}$$

$$\begin{aligned} \text{Example} \quad : F(s) &= \frac{2s + 3}{s^3 + 2s^2 + s} = \frac{2s + 3}{(s^2 + 2s + 1)s} \\ &= \frac{2s + 3}{(s + 1)^2 s} \\ &= \frac{k_1}{(s + 1)^2} + \frac{k_2}{(s + 1)} + \frac{k_3}{s} \end{aligned}$$

$$\text{Let } F_1(s) = (s + p_1)^r F(s)$$

$$\text{Then: } k_i = \frac{1}{(i-1)!} \left. \frac{d^{i-1} F_1(s)}{ds^{i-1}} \right|_{s \rightarrow -p_i} \quad i = 1, 2, \dots, r; \quad 0! = 1$$

Case 2: continue of the example $F(s)=(2s+3)/(s^3+2s^2+s)$.

$$F(s) = \frac{k_1}{(s+1)^2} + \frac{k_2}{(s+1)} + \frac{k_3}{s}$$

$$k_1 = (s+1)^2 F(s) \big|_{s=-1} = \frac{2s+3}{s} \big|_{s=-1} = -1$$

$$k_2 = \frac{1}{(2-1)!} \frac{d}{ds} [(s+1)^2 F(s)] \big|_{s=-1} = \frac{d}{ds} \left(\frac{2s+3}{s} \right) \big|_{s=-1}$$

$$k_2 = \frac{2s - (2s+3)(1)}{s^2} \big|_{s=-1} = \frac{-2-1}{1} = -3$$

$$k_3 = sF(s) \big|_{s=0} = \frac{2s+3}{(s+1)^2} \big|_{s=0} = 3$$

$$F(s) = \frac{-1}{(s+1)^2} + \frac{-3}{s+1} + \frac{3}{s}$$

$$f(t) = 3 - 3e^{-t} - te^{-t}$$

Case 3: Roots of the Denominator are Complex.

Example: $F(s)=10/(s^3+4s^2+9s+10)$

$$F(s) = \frac{10}{s^3 + 4s^2 + 9s + 10} = \frac{10}{(s + 2)[(s + 1)^2 + 2^2]}$$

$$F(s) = \frac{k_1}{s + 2} + \frac{k_2}{s + 1 + j2} + \frac{k_2^*}{s + 1 - j2}$$

$$k_1 = (s + 2)F(s) \big|_{s=-2} = \frac{10}{(s + 1)^2 + 4} \big|_{s=-2} = 2$$

$$k_2 = (s + 1 + j2)F(s) = \frac{10}{(s + 2)(s + 1 - j2)} \big|_{s=-1-j2}$$

$$k_2 = \frac{10}{(-1 - j2 + 2)(-1 - j2 + 1 - j2)} = \frac{10}{(1 - j2)(-j4)}$$

$$k_2 = \frac{10}{(2.236 \angle -63.4^\circ)(4 \angle -90^\circ)} = 1.118 \angle 153.4^\circ$$

$$f(t) = 2e^{-2t} + 2.23 e^{-t} \cos(2t - 153.4^\circ)$$

$$f(t) = 2e^{-2t} - 2e^{-t} \cos(2t) + e^{-t} \sin(2t)$$

For the Tutorial next week, use Matlab to find the Inverse Laplace Transform for the case where the roots of the denominator are complex roots.

The Transfer Function

The n th - order differential equation :

$$\frac{d^n c}{dt^n} + a_{n-1} \frac{d^{n-1} c}{dt^{n-1}} + \dots + a_1 \frac{dc}{dt} + a_0 c = b_m \frac{d^m r}{dt^m} + b_{m-1} \frac{d^{m-1} r}{dt^{m-1}} + \dots + b_1 \frac{dr}{dt} + b_0 r$$

Laplace transform :

$$(s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0)C(s) = (b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0)R(s)$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$$

The characteristic equation is defined as :

$$s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0$$

The roots of this equation are called *poles* of the system.

The roots of the equation :

$$b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0 = 0$$

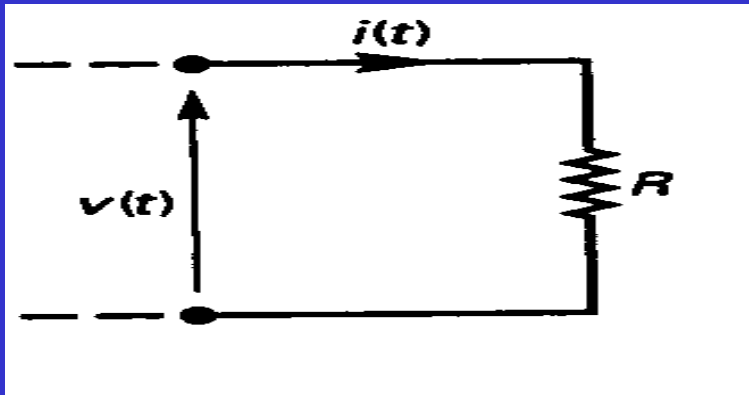
are called *zeros* of the system.

Transfer Function for a differential equation

Exercise: Do examples 2.4 and 2.5 on the Textbook.

Models of Electrical Circuits

- Resistance circuit: $v(t) = i(t) R$

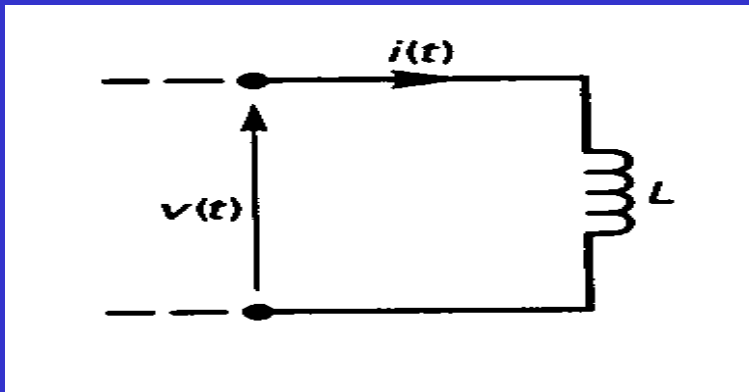


$$v(t) = i(t) R$$

$$V(s) = I(s) R$$

$$\frac{V(s)}{I(s)} = R$$

- Inductance circuit:



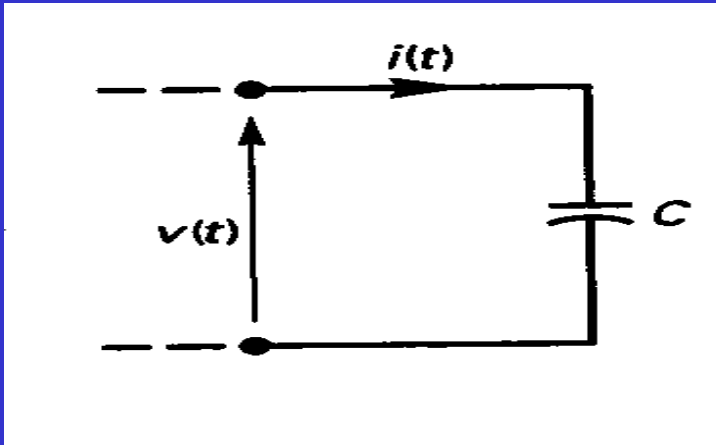
$$v(t) = L \frac{di(t)}{dt}$$

$$V(s) = L(sI(s) - i(0))$$

$$\frac{V(s)}{I(s)} = Ls \quad \text{if} \quad i(0) = 0$$

Models of Electrical Circuits

- Capacitance circuit:



$$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau + v(0)$$




$$V(s) = \frac{1}{C} \frac{I(s)}{s} + \frac{v(0)}{s}$$

$$\frac{V(s)}{I(s)} = \frac{1}{Cs} \quad \text{if } v(0) = 0$$

$$\frac{\ell(\text{output})}{\ell(\text{input})} = \text{Transfer function}$$

Transfer Functions for Electrical Networks

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

| Component | Voltage-current | Current-voltage | Voltage-charge | Impedance $Z(s) = V(s)/I(s)$ | Admittance $Y(s) = I(s)/V(s)$ |
|--|---|---|----------------------------------|---------------------------------|----------------------------------|
|  Capacitor | $v(t) = \frac{1}{C} \int_0^1 i(\tau) d\tau$ | $i(t) = C \frac{dv(t)}{dt}$ | $v(t) = \frac{1}{C} q(t)$ | $\frac{1}{Cs}$ | Cs |
|  Resistor | $v(t) = Ri(t)$ | $i(t) = \frac{1}{R} v(t)$ | $v(t) = R \frac{dq(t)}{dt}$ | R | $\frac{1}{R} = G$ |
|  Inductor | $v(t) = L \frac{di(t)}{dt}$ | $i(t) = \frac{1}{L} \int_0^1 v(\tau) d\tau$ | $v(t) = L \frac{d^2 q(t)}{dt^2}$ | Ls | $\frac{1}{Ls}$ |

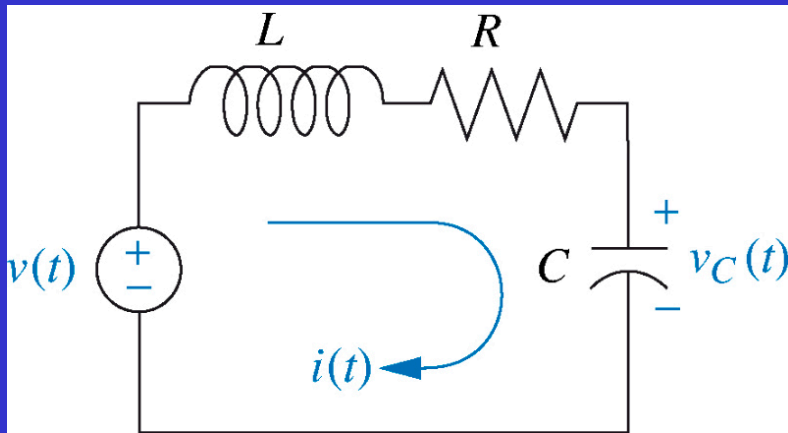
- Kirchhoff's voltage law:

The algebraic sum of voltages around any closed loop in an electrical circuit is zero.

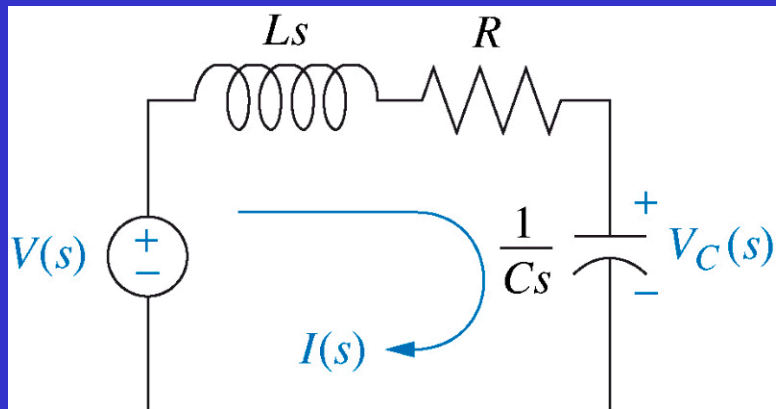
- Kirchhoff's current law:

The algebraic sum of currents into any junction in an electrical circuit is zero.

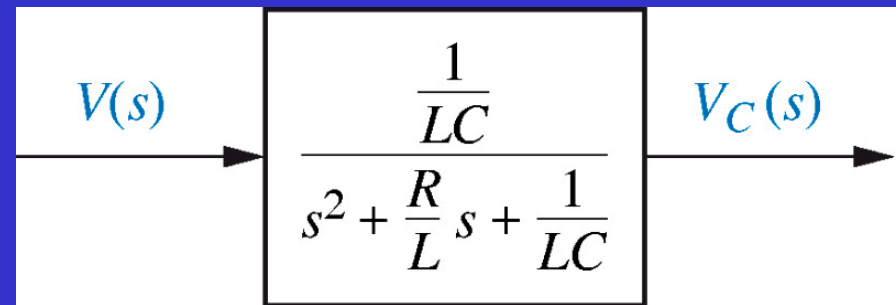
Electrical network (Example 2.6)



Original network

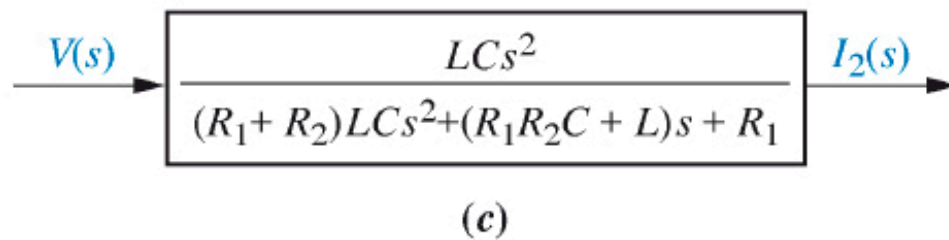
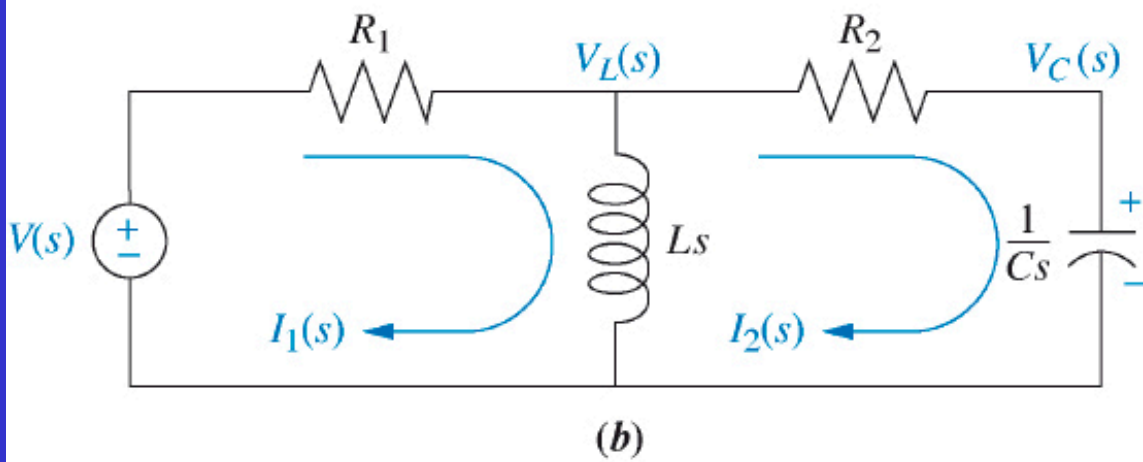
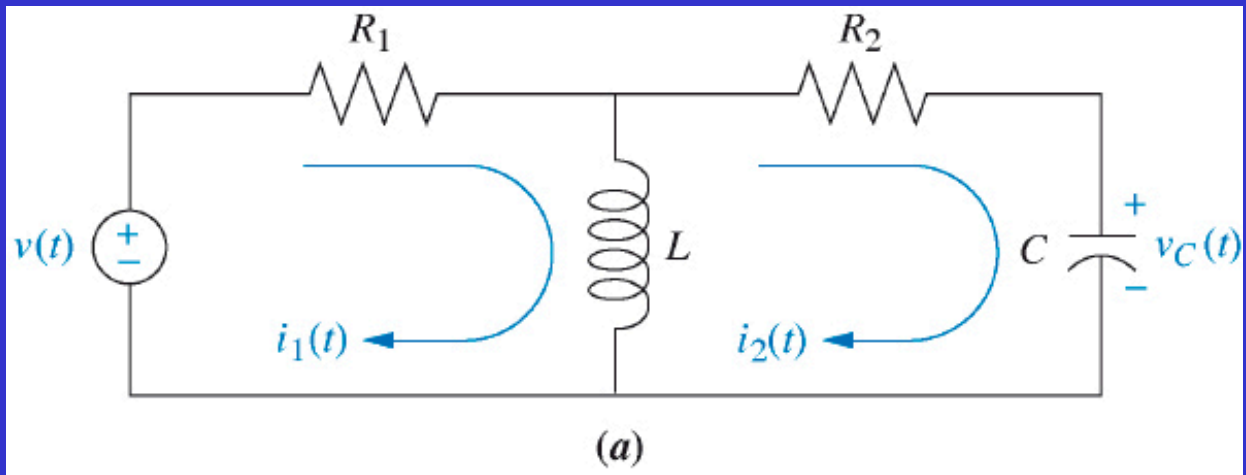


Laplace Transform network



Transfer Function network

- Exercise: Do example 2.10 of the textbook
- Excluding Operational Amplifiers



fig_02_06

Models of Mechanical Systems

Mechanical translational systems.

- Newton's second law:

$$f(t) = Ma(t) = M \frac{dv(t)}{dt} = M \frac{d^2x(t)}{dt^2}$$

- Device with friction (shock absorber):

$$f(t) = B \left[\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt} \right]$$

B is damping coefficient.

- Translational system to be defined is a spring (Hooke's law):

$$f(t) = K[x_1(t) - x_2(t)]$$

K is spring coefficient

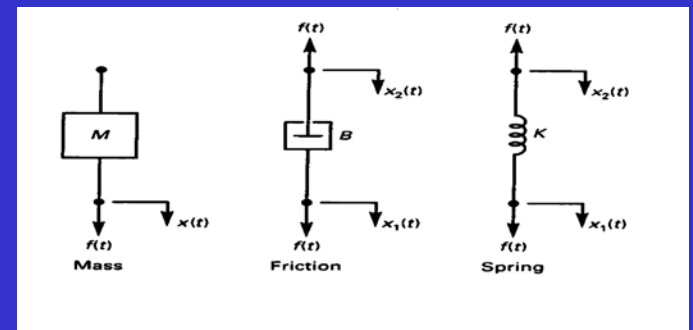
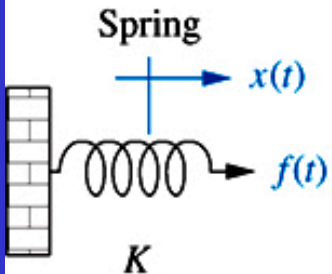
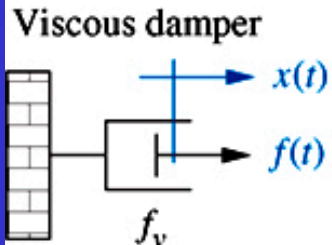
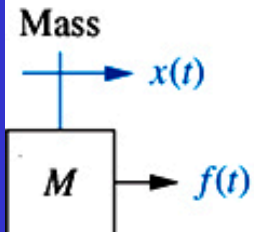
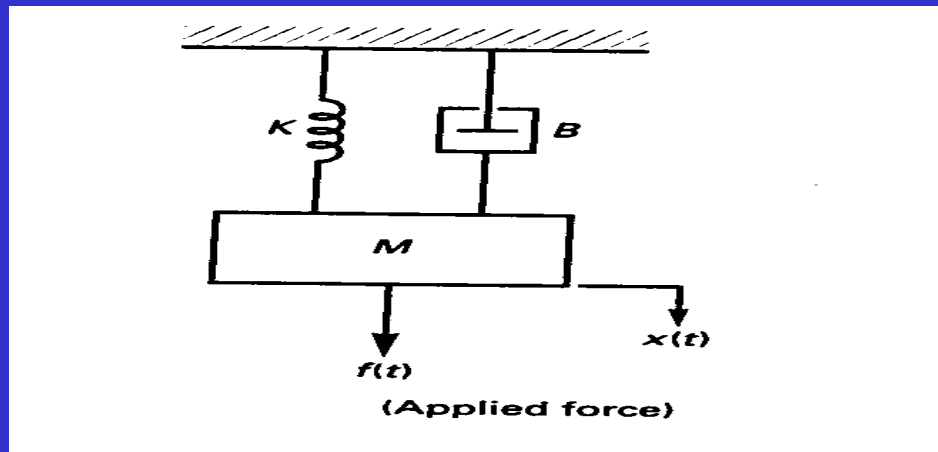


TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

| Component | Force-velocity | Force-displacement | Impedance $Z_M(s) = F(s)/X(s)$ |
|--|-----------------------------------|----------------------------------|-----------------------------------|
| <p>Spring</p>  | $f(t) = K \int_0^t v(\tau) d\tau$ | $f(t) = Kx(t)$ | K |
| <p>Viscous damper</p>  | $f(t) = f_v v(t)$ | $f(t) = f_v \frac{dx(t)}{dt}$ | $f_v s$ |
| <p>Mass</p>  | $f(t) = M \frac{dv(t)}{dt}$ | $f(t) = M \frac{d^2 x(t)}{dt^2}$ | $M s^2$ |

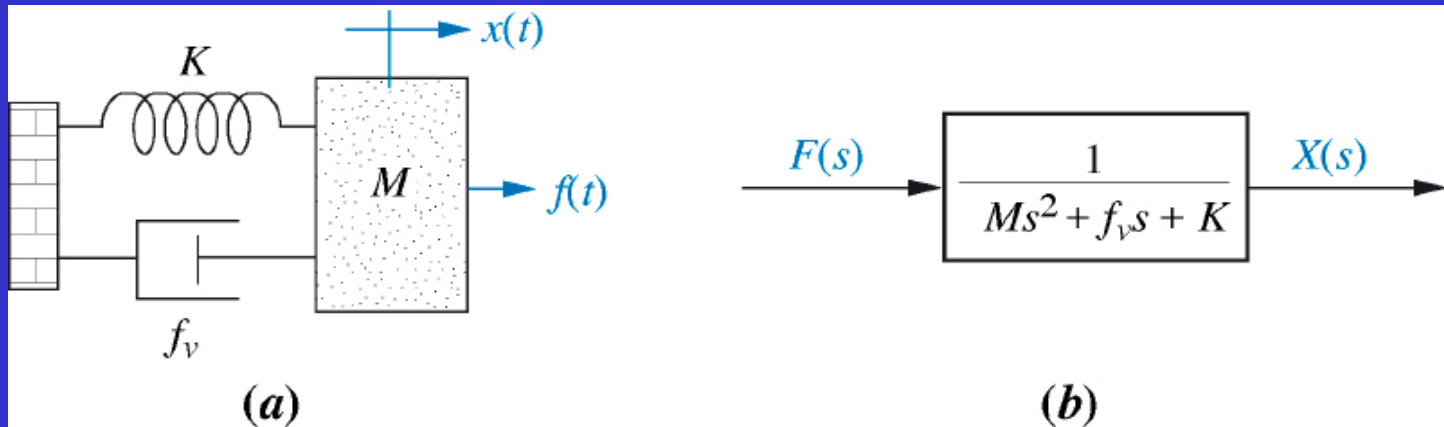
- Model of a mass-spring-damper system:

$$M \frac{d^2 x(t)}{dt^2} + B \frac{dx(t)}{dt} + Kx(t) = f(t)$$

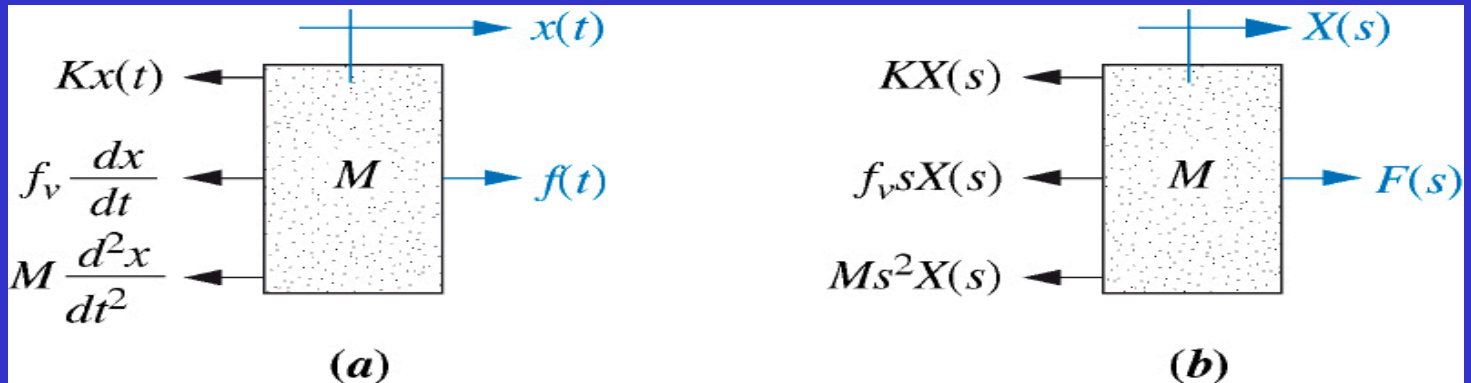


- Note that linear physical systems are modeled by linear differential equations for which linear components can be added together. See example of a mass-spring-damper system.

Transfer Functions for Mechanical Systems



$$M \frac{d^2 x(t)}{dt^2} + B \frac{dx(t)}{dt} + Kx(t) = f(t) \quad \Rightarrow \quad Ms^2 X(s) + BsX(s) + KX(s) = F(s)$$



Exercise: Do example 2.17 of the text

The idea is to set up the equations in the form :

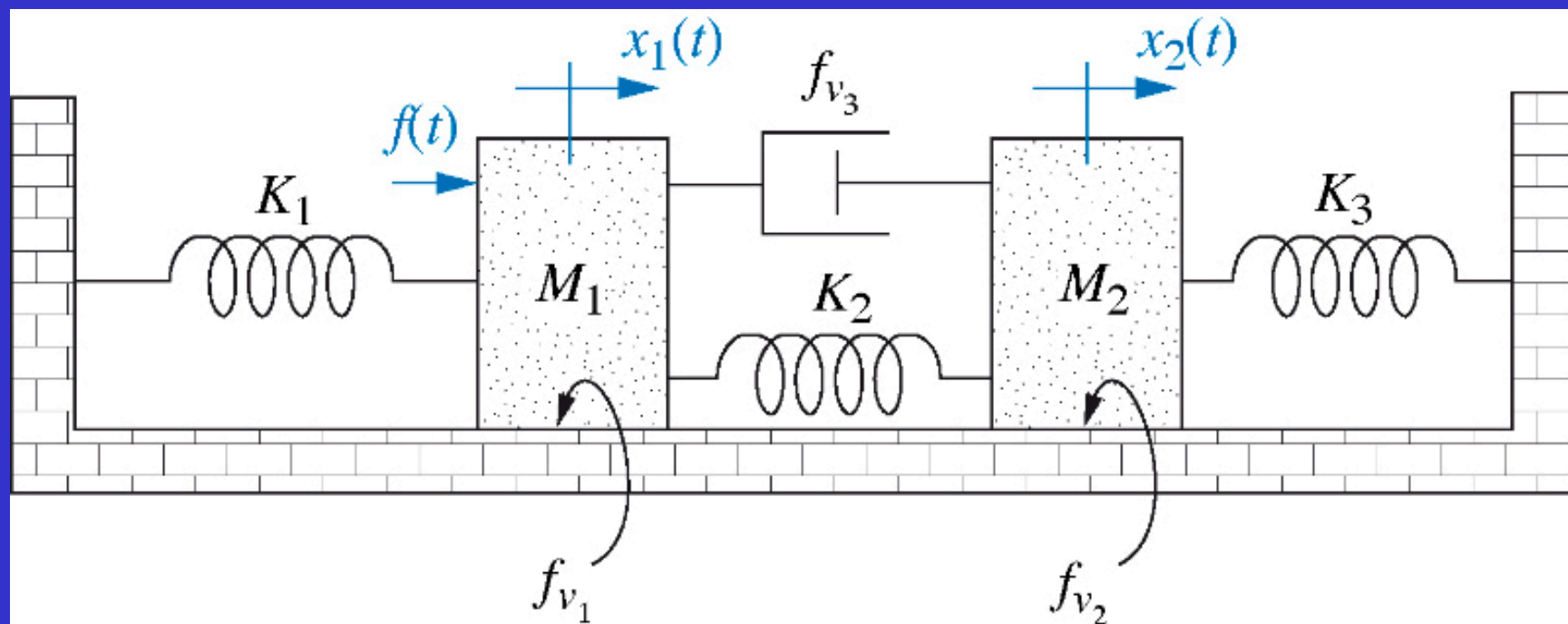
$$ax_1 + bx_2 = \alpha$$

$$cx_1 + dx_2 = \beta$$

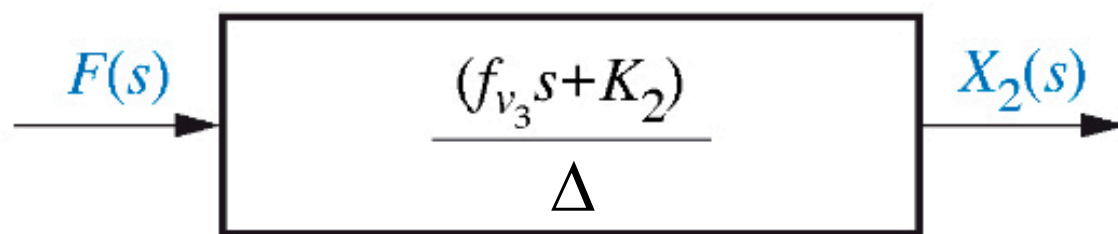
Using Cramer's rule, the solutions are then :

$$x_1 = \frac{\Delta_1}{\Delta}; \quad x_2 = \frac{\Delta_2}{\Delta}$$

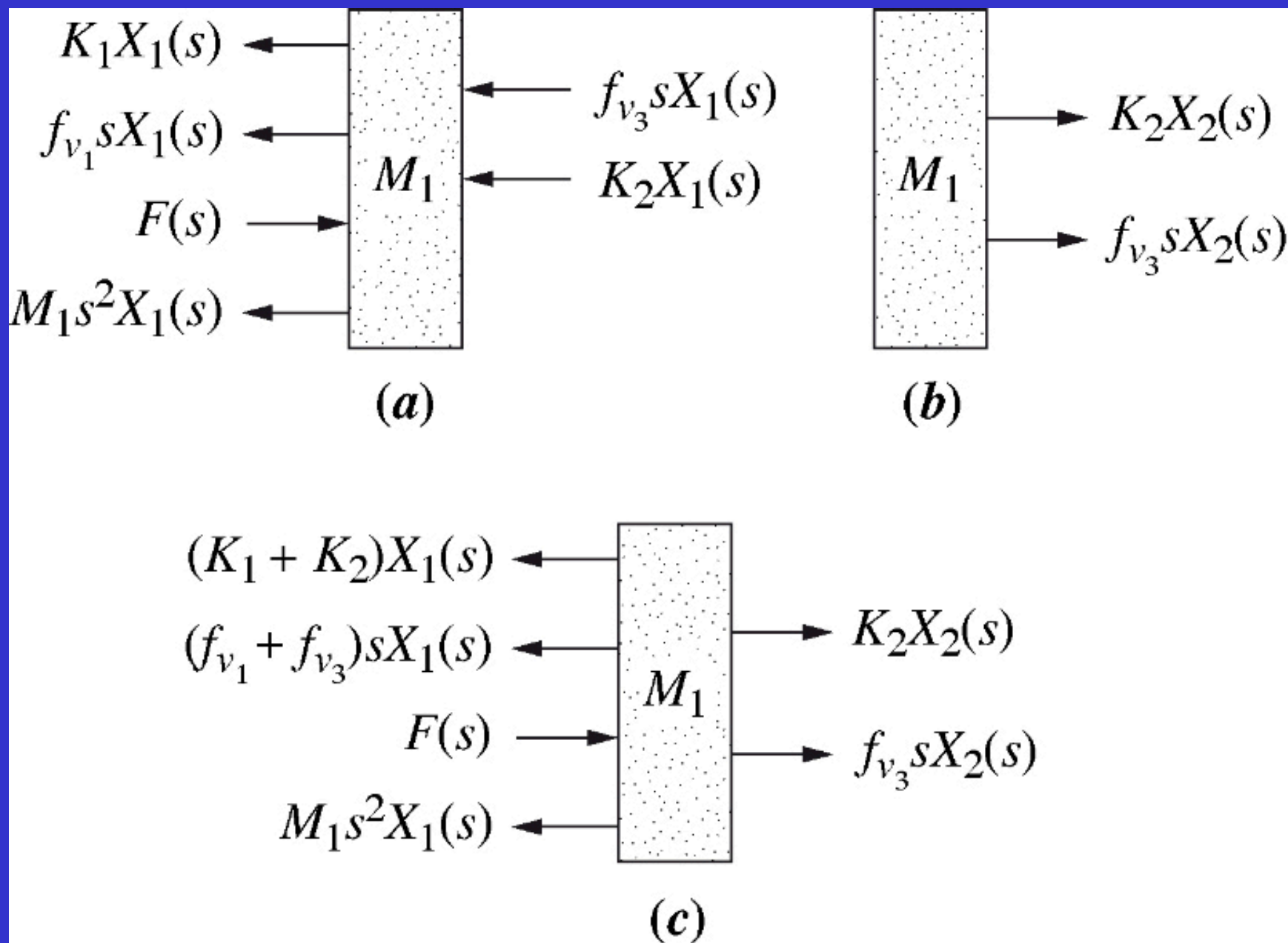
$$\text{where : } \Delta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \Delta_1 = \begin{bmatrix} \alpha & b \\ \beta & d \end{bmatrix}; \Delta_2 = \begin{bmatrix} a & \alpha \\ c & \beta \end{bmatrix}$$

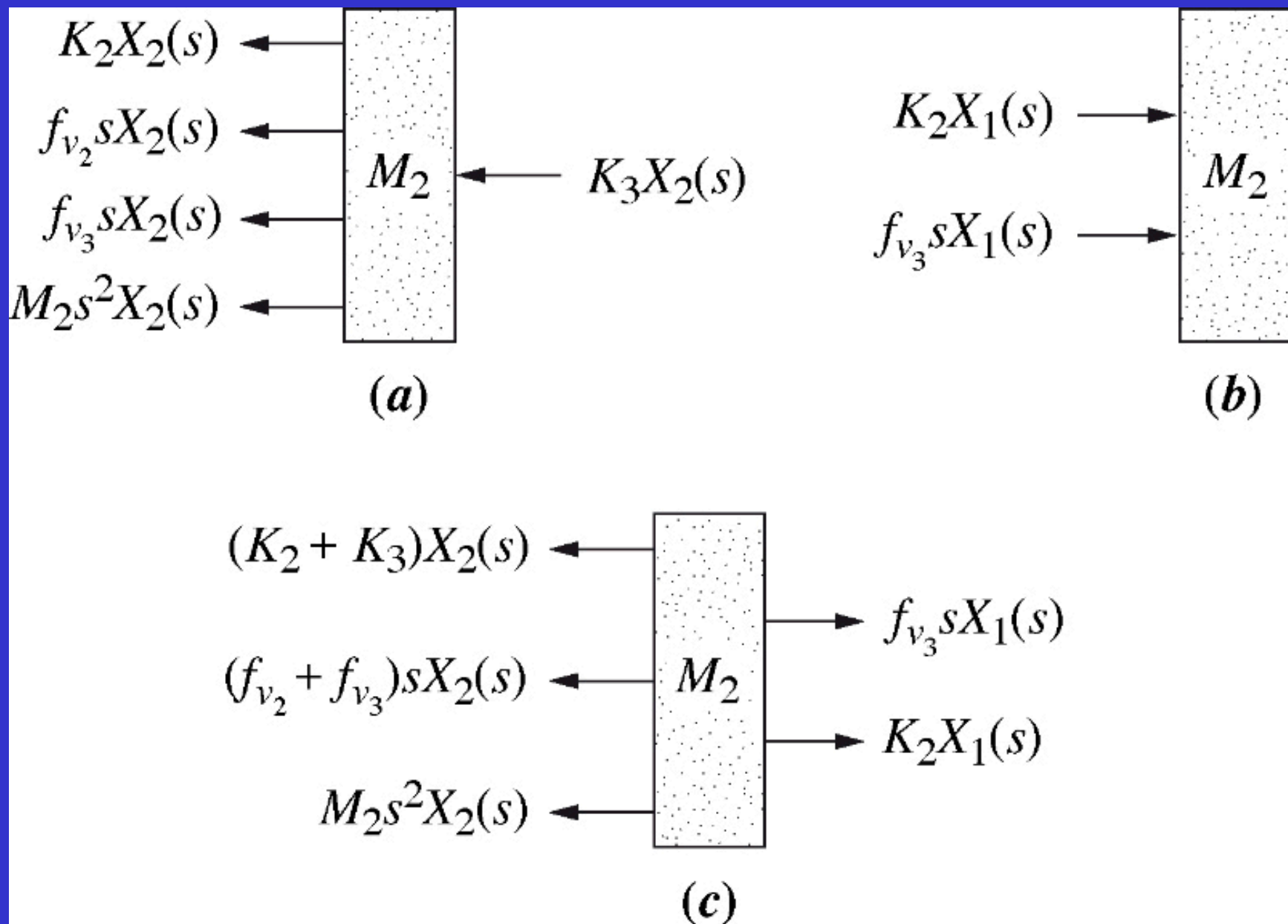


(a)



(b)

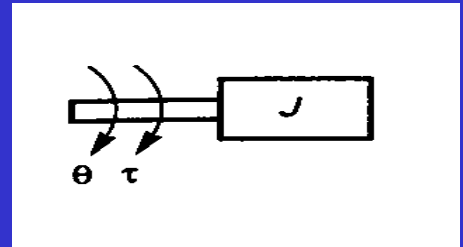




Mechanical Rotational Systems

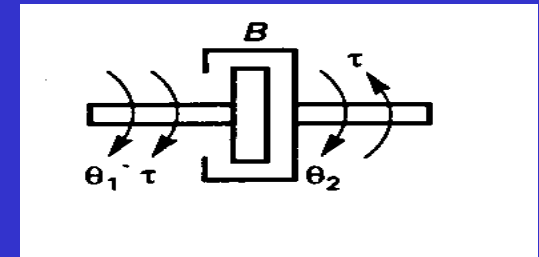
- Moment of inertia:

$$\tau(t) = J \frac{d^2 \theta(t)}{dt^2} \quad \tau(s) = Js^2 \theta(s)$$



- Viscous friction:

$$\tau(t) = B \left(\frac{d\theta(t)}{dt} \right) \quad \tau(s) = Bs\theta(s)$$



- Torsion:

$$\tau(t) = K[\theta(t)] \quad \tau(s) = K[\theta(s)]$$

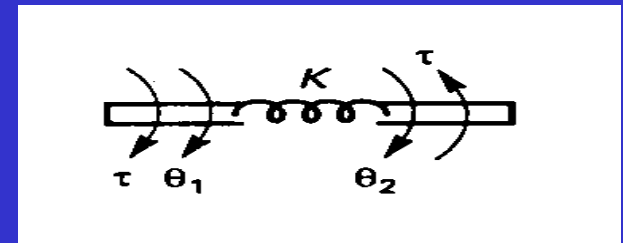
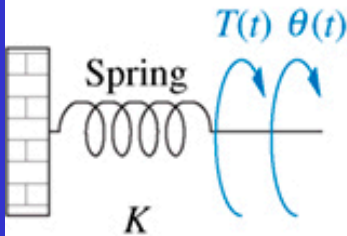
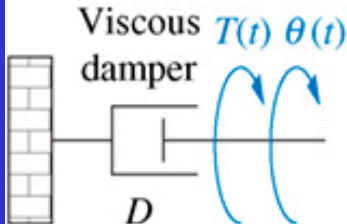
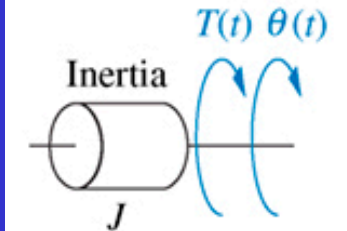


TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

| Component | Torque-angular velocity | Torque-angular displacement | Impedance $Z_M(s) = T(s)/\theta(s)$ |
|--|--|--------------------------------------|--|
|  | $T(t) = K \int_0^t \omega(\tau) d\tau$ | $T(t) = K\theta(t)$ | K |
|  | $T(t) = D\omega(t)$ | $T(t) = D \frac{d\theta(t)}{dt}$ | Ds |
|  | $T(t) = J \frac{d\omega(t)}{dt}$ | $T(t) = J \frac{d^2\theta(t)}{dt^2}$ | Js^2 |

- Model of a torsional pendulum (pendulum in clocks inside glass dome):

Moment of inertia of pendulum bob denoted by J

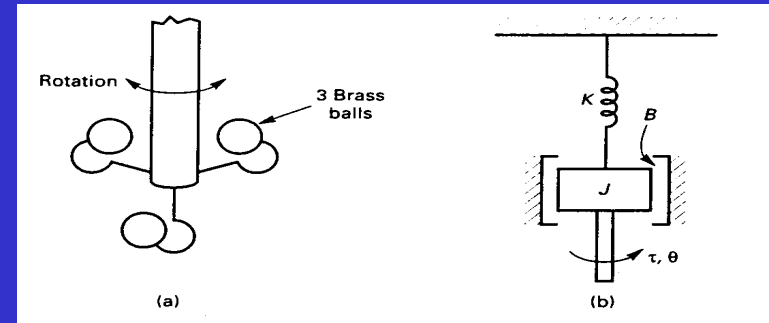
Friction between the bob and air by B

Elastance of the brass suspension strip by K

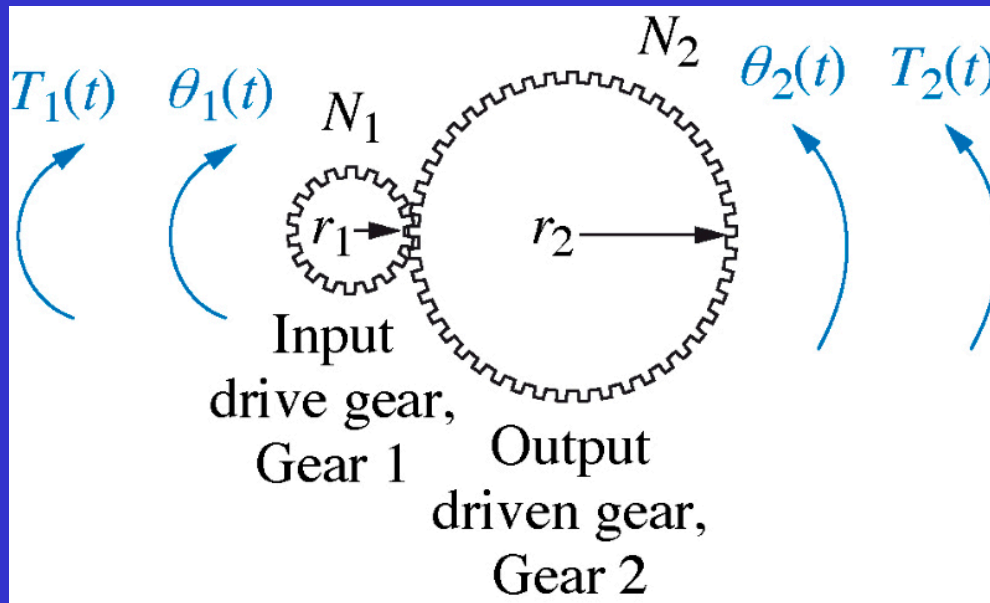
$$J \frac{d^2 \theta(t)}{dt^2} + B \frac{d\theta(t)}{dt} + K \theta(t) = \tau(t)$$

$$Js^2 \Theta(s) + Bs \Theta(s) + K \Theta(s) = \tau(s)$$

$$\frac{\Theta(s)}{\tau(s)} = \frac{1}{Js^2 + Bs + K} = \text{Transfer function}$$



Transfer Functions for Systems with Gears

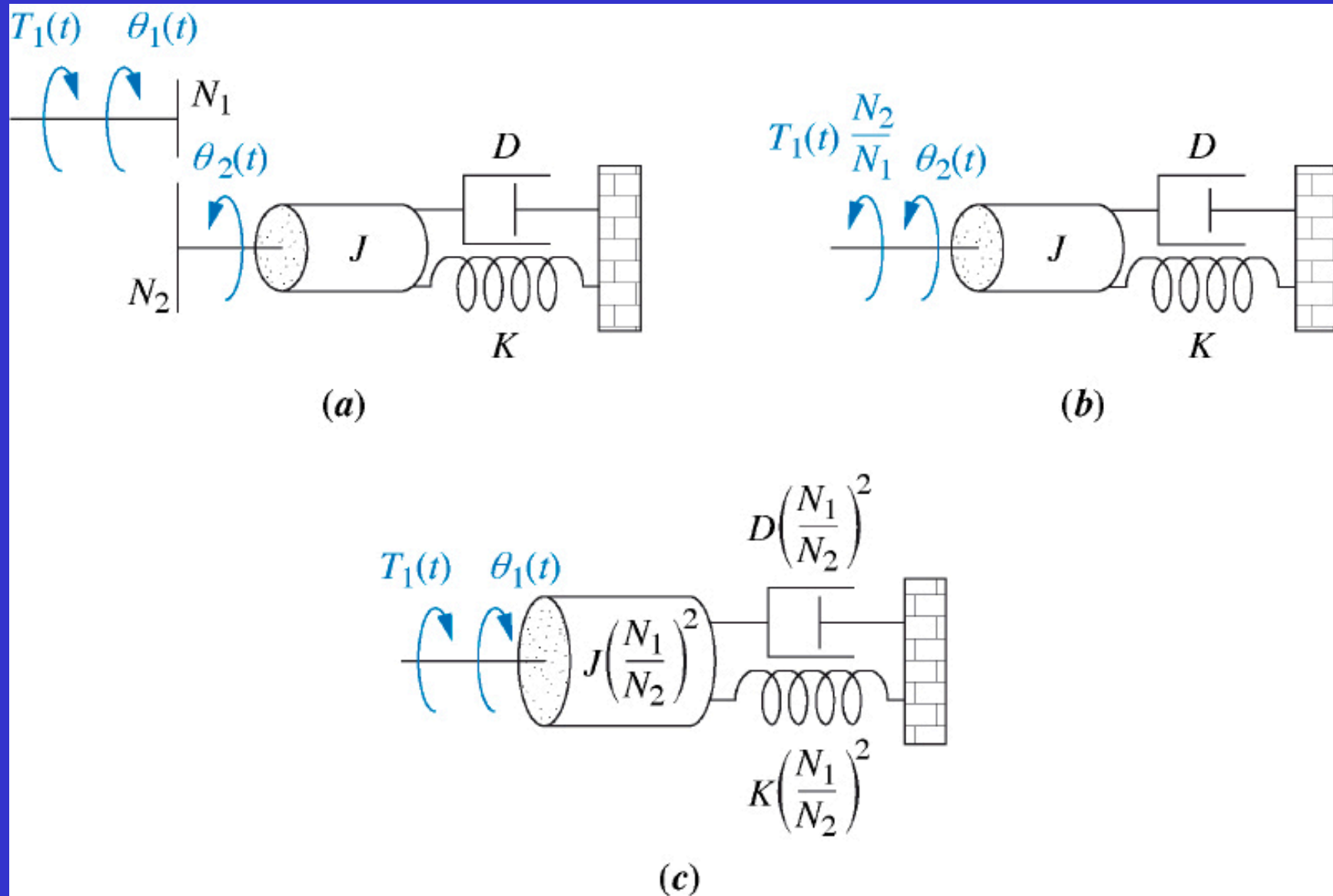


$$r_1\theta_1 = r_2\theta_2$$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

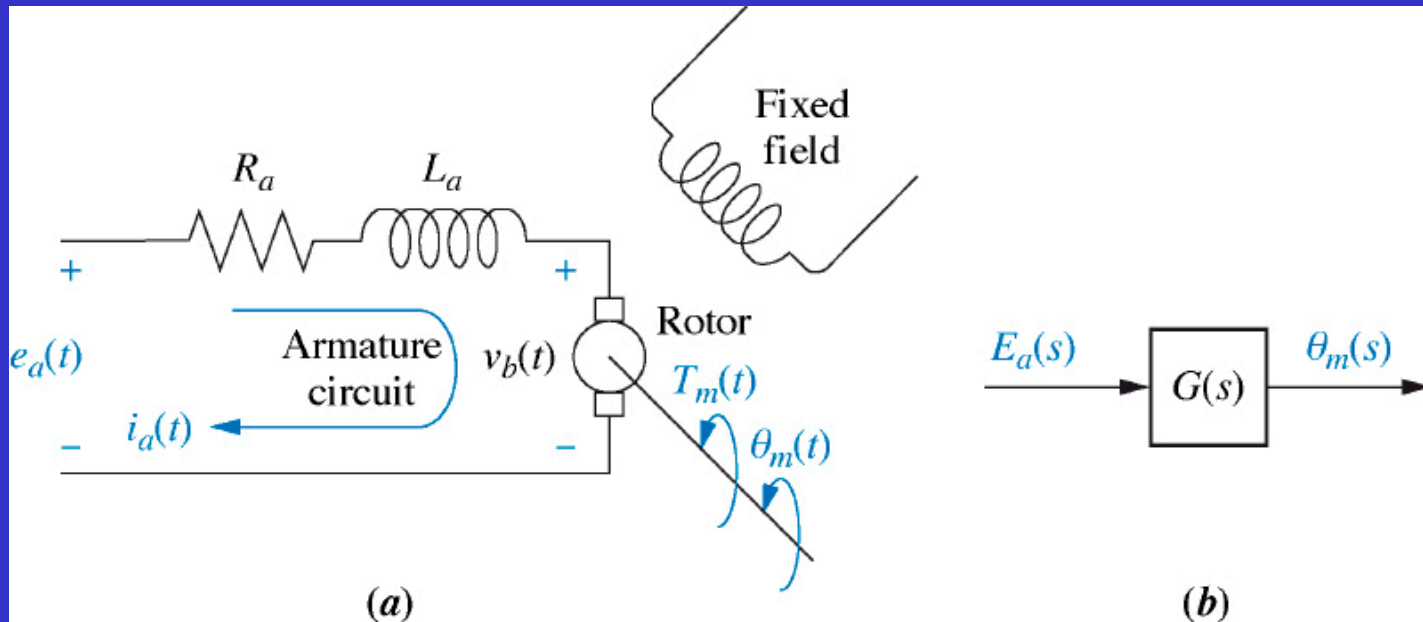
$$T_1\theta_1 = T_2\theta_2$$

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

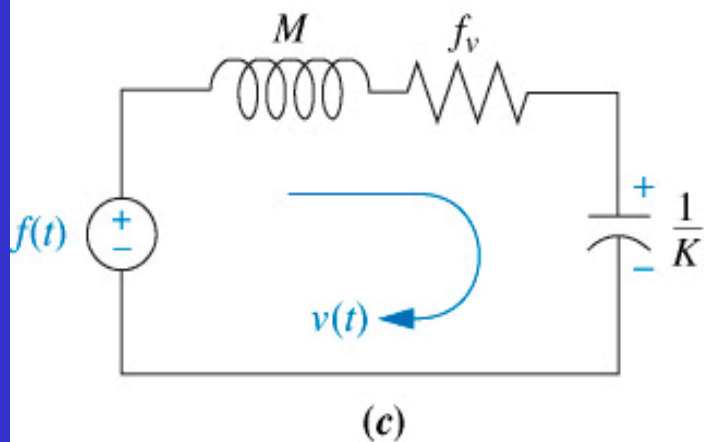
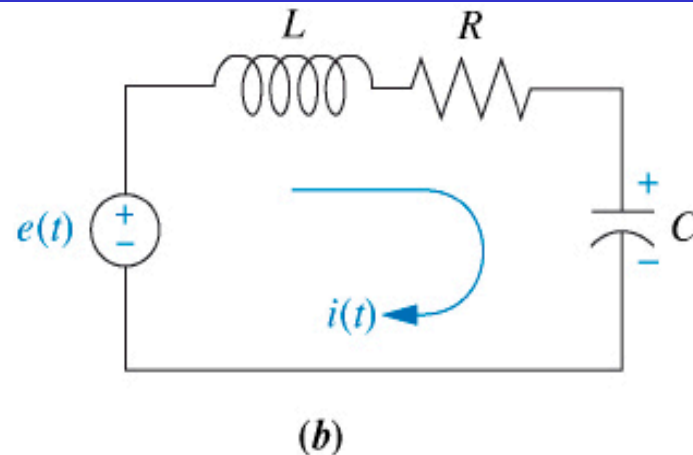
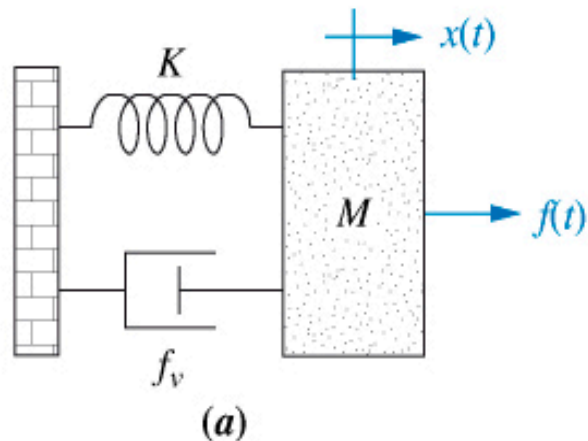


fig_02_29

Electro-mechanical System Transfer Functions



Equivalence between Mechanical and Electrical Systems



| | | |
|------------------------|------------------|------------------------|
| mass = M | → inductor | = M henries |
| viscous damper = f_v | → resistor | = f_v ohms |
| spring = K | → capacitor | = $\frac{1}{K}$ farads |
| applied force = $f(t)$ | → voltage source | = $f(t)$ |
| velocity = $v(t)$ | → mesh current | = $v(t)$ |

(d)

- Differential equations as mathematical models of physical systems: similarity between mathematical models of electrical circuits and models of simple mechanical systems (see model of an RCL circuit and model of the mass-spring-damper system).

$$Ri(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt = E(t)$$

$$M \frac{d^2 x(t)}{dt^2} + B \frac{dx(t)}{dt} + Kx(t) = f(t)$$

Kirchhoffs and Newton's laws lead to mathematical models that describe the relationship between the input and output of dynamic system. One such model is the time invariant, n th-order differential equation:

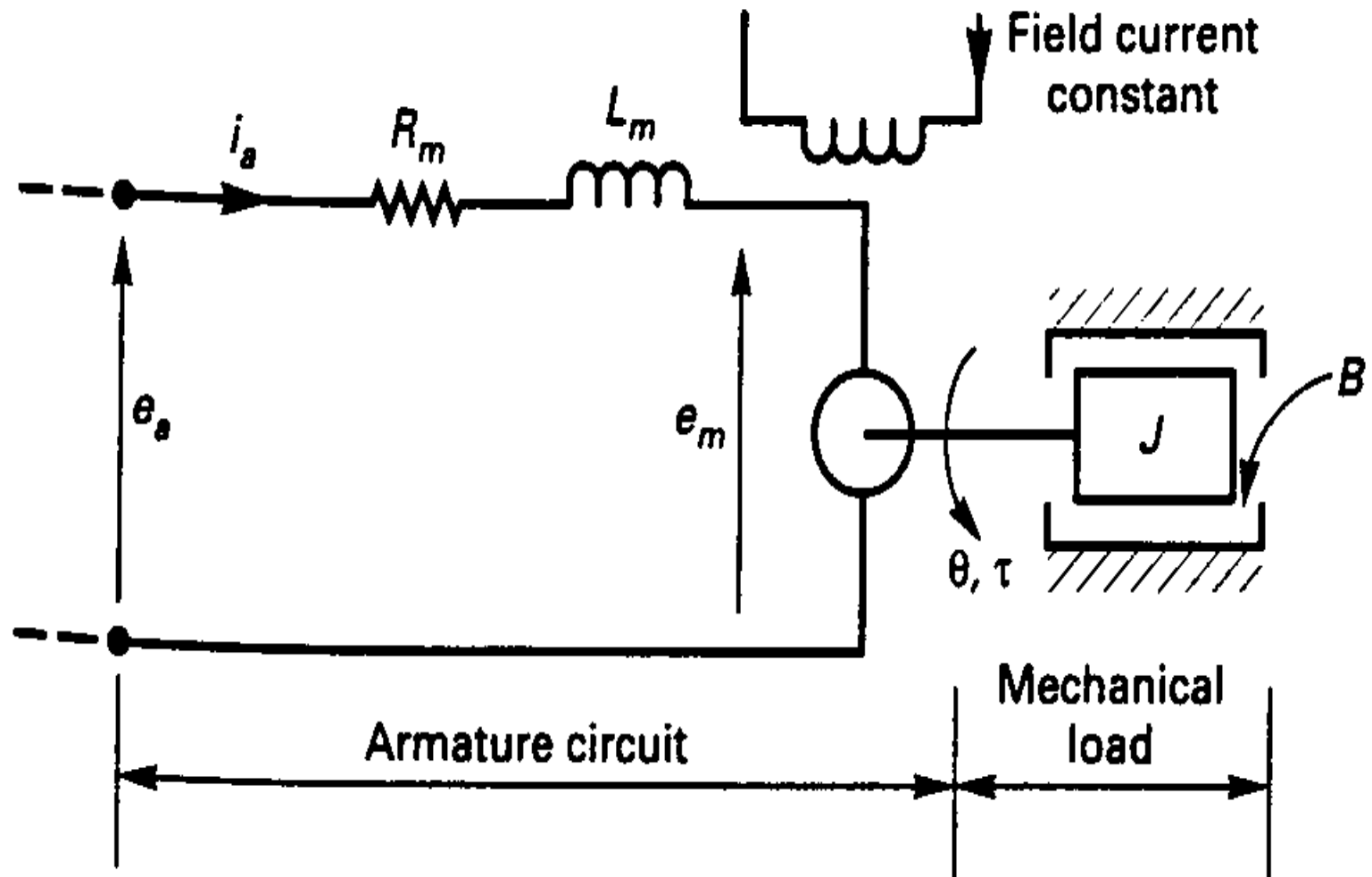
$$\frac{d^n c}{dt^n} + a_{n-1} \frac{d^{n-1} c}{dt^{n-1}} + \dots + a_1 \frac{dc}{dt} + a_0 c = b_m \frac{d^m r}{dt^m} + b_{m-1} \frac{d^{m-1} r}{dt^{m-1}} + \dots + b_1 \frac{dr}{dt} + b_0 r$$

where $c(t)$ is the output, and $r(t)$ is the input.

This leads to the concept of the Transfer Function which is the corner stone of the control theory. Transfer functions are defined from the Laplace Transform.

- Summary of through- and across-variables for physical systems:

| System | Variable Through Element | Integrated Through Variable | Variable Across Element | Integrated Across Variable |
|--------------------------|------------------------------------|-----------------------------|--|--|
| Electrical | Current, i | Charge, q | Voltage difference, v_{21} | Flux linkage, λ_{21} |
| Mechanical Translational | Force, F | Translational momentum, P | Velocity difference, v_{21} | Displacement difference, y_{21} |
| Mechanical rotational | Torque, T | Angular momentum, h | Angular velocity difference, ω_{21} | Angular displacement difference, θ_{21} |
| Fluid | Fluid volumetric rate of flow, Q | Volume, V | Pressure difference, P_{21} | Pressure momentum, γ_{21} |
| Thermal | Heat flow rate, q | Heat energy, H | Temperature difference, τ_{21} | |



- Model of electromechanical systems.
- Model of a servomotor:

$$i_a(t) = \frac{1}{R_m} (e_a(t) - e_m(t))$$

$$e_m(t) = K_m \frac{d\theta}{dt}$$

$$\tau(t) = \frac{K_\tau}{R_m} (e_a(t) - e_m(t))$$

$$J \frac{d^2\theta(t)}{dt^2} = \tau(t) - B \frac{d\theta(t)}{dt}$$

$$J \frac{d^2\theta(t)}{dt^2} + B \frac{d\theta(t)}{dt} = k_1 e_a(t) - k_2 \frac{d\theta}{dt}$$

- Model of temperature-control system:

$q_e(t)$ = heat flow given by electric heater

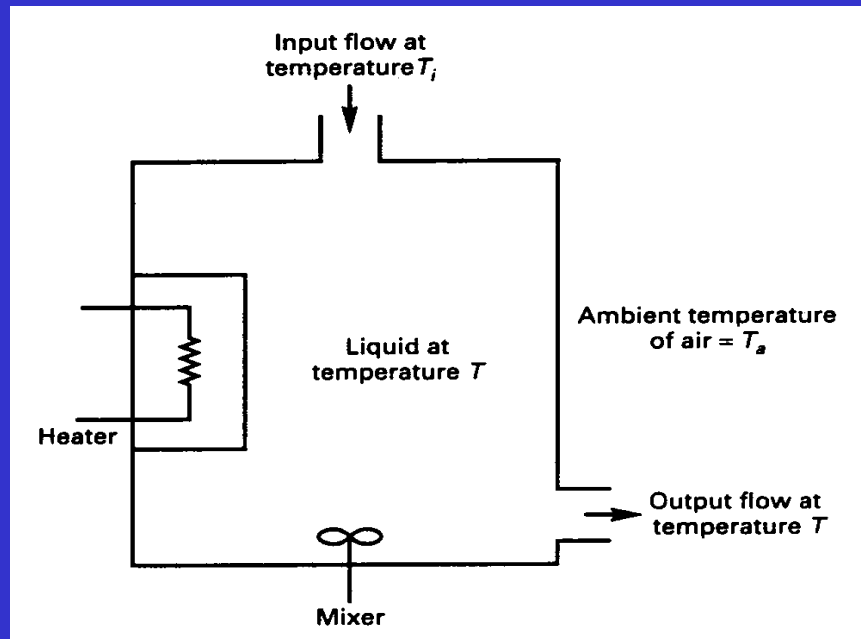
$q_i(t)$ = heat flow via liquid entering tank $= VHT_i$

$q_l(t)$ = heat flow into liquid $= C \frac{dT}{dt}$

$q_o(t)$ = heat flow via liquid leaving tank $= VHT$

$q_s(t)$ = heat flow (loss) via tank surface $= \frac{T - T_a}{R}$

$$q_e(t) + q_i(t) = q_l(t) + q_o(t) + q_s(t)$$



H = specific heat of liquid

C = thermal capacity

R = thermal resistance

V = volume of liquid

$$q_e + VHT_i = C \frac{dT}{dt} + VHT + \frac{T - T_a}{R}$$